Coalgebraic Analysis of Social Systems

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Abstract

The algebraic analysis of social systems, or algebraic social network analysis, refers to a collection of methods designed to extract information about the structure of social systems modeled as directed graphs. These methods focus on identifying the roles within a system, understood as compound relations, as well as the social positions, understood as groups of similarly connected actors. The primary focus is on pairwise relationships between social actors, modeled through directed graphs.

Higher-order relationships in social systems have received considerable attention in recent years. In this work, we use the theory of universal coalgebra to formalize these notions for graphs and extend them to models of social systems that also account for higher-order interactions between social actors, such as hypergraphs. Finally, we unify the analysis of social roles through a functoriality theorem.

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Algebraic social network analysis uses rigorous algebraic methods to extract information about the structure of social networks, modelled as *multirelational directed graphs* [7] consisting of vertices and multiple binary relations. A central early problem in algebraic social network analysis was determining the appropriate notion of '*social position*' within a network. Roughly speaking, one would like to say that two actors share the same social position if they are *similar* in their connections. Many definitions of similarity have been proposed, including structural [6] and automorphic equivalence [9]. In the late 1970s, structural and automorphic equivalence were superseded by the substantially more general concept of *regular equivalence* [8], under which actors are similar if they are 'similarly related to similar actors'.

Another early problem was to investigate *social roles* actors hold towards each other. These roles are given by the individual relations in a network (such as 'being a friend of'), as well as the compound roles obtained by composing them (such as 'being a friend of an enemy of'). The key insight of algebraic social network analysis is that one can obtain insight about social networks by studying the semigroups generated by composing roles.

These analyses are traditionally only considered for social systems modelled as multirelational graphs. Real-world social systems, however, often contain higher-order relations involving groups of actors, rather than only pairs as in graphs. Recent increased interest in higher-order models of social systems [3], motivates extending the analyses accordingly.

In this abstract we present **our contribution** to generalising positional and role analysis beyond graphs. At the heart of our work is the observation that regular equivalence is an instance of *bisimulation equivalence*, central in the study of transition systems and modal logic. Building on this, we use the theory of *universal coalgebra* as it is a natural framework for generalizing graphs and regular equivalences. The coalgebraic method not only generalizes the two analyses, but also clarifies their connection via a theorem stating that role analysis is *functorial* with respect to positional analysis.

We start from a category $\mathbf{C}$ and an endofunctor $T\colon \mathbf{C} \to \mathbf{C}$ for which we look at the T-coalgebras as our models of social systems. The case of graphs is recovered by taking (in the category of sets) the powerset functor. Taking the double powerset functor $\mathcal{PP}$ gives us *directed hypergraphs* [1], serving as our prototypical example of a higher-order model.

As per our observation identifying regular equivalences as bisimulations for graphs, we base our generalisation of positional analysis on T-bisimulation equivalences. In positional analysis, similarity relations provide *positional reductions*: surjective functions mapping actor to their social position, consisting of all actors considered similar to them. In the coalgebraic setting, when considering so-called *multirelational T-coalgebras*, these positional reductions correspond precisely the epimorphic multirelational T-coalgebra homomorphisms, thus allowing us to generalise positional analysis based on regular equivalences beyond graphs to, e.g., directed hypergraphs. In role analysis, the main object of study is the semigroup generated by composing the relations in a multirelational graph. To generalise this to T-coalgebras, a natural approach would be to require a notion of composing a T-coalgebra with itself. When T has the structure of a monad, it is well-known that this can be done by considering the coalgebra to be an endomorphism in the Kleisli category of T. Unfortunately, [5] showed that $\mathcal{PP}$ is not a monad, meaning our prototypical example of directed hypergraphs as $\mathcal{PP}$-coalgebras fails. Luckily, as noted by [2], $\mathcal{PP}$ *does* form a *semimonad* (monad without unit), aligning perfectly with our goal of generating semigroups rather than monoids.

Putting both analyses together, our main result is that role analysis based on a semimonad T forms a functor from multirelational T-coalgebras and epimorphic homomorphisms (corresponding to positional reductions) to semigroups and surjective semigroup homomorphisms.

We believe our work opens exciting theoretical and practical directions for generalising role and positional analysis beyond graphs. We already have generalisations of the analyses beyond coalgebras using enriched *semipromonads*, semimonads of enriched profunctors, allowing functorial role analysis with e.g. role *quantales* rather than semigroups. We will also explore the generalisation of the notion of approximate similarity between actors as used in algebraic social network analysis, to the coalgebraic setting via behavioural metrics [4] and apply the methods to the study of real-world social systems exhibiting higher-order relations.

References

1 G. Ausiello and L. Luigi. Directed hypergraphs: Introduction and fundamental algorithms—a survey. *Theoretical Computer Science*, 2017.

2 J. Baez and G. Egan. Discussion on The n-Category Café, June 2018. URL: `https://golem.ph.utexas.edu/category/2018/06/sets_of_sets_of_sets_of_sets_o.html#c054106`.

3 C. Bick et al. What are higher-order networks? *SIAM Review*, 2023.

4 K. D'Angelo et al. Behavioural Metrics: Compositionality of the Kantorovich Lifting and an Application to Up-To Techniques. In *Proc. of CONCUR 2024*.

5 B. Klin et al. Iterated covariant powerset is not a monad. In *Proc. of the 34th Conference on the Mathematical Foundations of Programming Semantics*, 2018.

6 F. Lorrain and H. C. White. Structural equivalence of individuals in social networks. *Journal of Mathematical Sociology*, 1971.

7 P. Pattison. *Algebraic Models for Social Networks*. Structural Analysis in the Social Sciences. Cambridge University Press, 1993.

8 L. D. Sailer. Structural equivalence: Meaning and definition, computation and application. *Social Networks*, 1978.

9 C. Winship. Thoughts about roles and relations: An old document revisited. *Social Networks*, 1988.