Extending Blockmodel Analysis to Higher-Order Models of Social Systems

Nima Motamed
Utrecht University

Joint work in progress with Daniel Cicala, Rachel Hardeman Morrill, Nikola Milećević, Nina Otter, and Emily Roff

JMM 2023
January 2023

This material is based upon work supported by the National Science Foundation under Grant Number DMS 1641020.

Plan for today

I. Blockmodel analysis for networks

II. Blockmodel analysis through coalgebra

III. Relating blockmodel and role analysis through coalgebra

Part I

Blockmodel analysis for networks

The idea

Study **social positions**: collections of actors who are similar to each other, usually in terms of their ties.

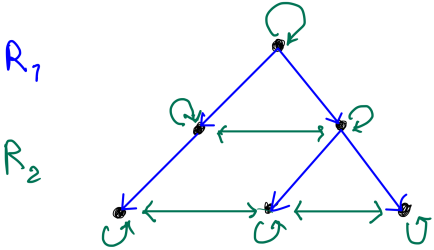
k-graphs

Definition

A ***k*-graph** consists of a (finite) set V of vertices and a family $R_1, \ldots, R_k$ of binary relations on V.

Example

In the 2-graph on the right, perhaps the vertices are employees in a firm, R_1 is the 'boss of' relation and R_2 is the 'collaborator of' relation.



This running example is from Otter & Porter (2020).

Basics of blockmodel analysis

Definition(ish) The aim of **blockmodel analysis** is to identify a partition $\overline{V}$ of V whose elements are **positions**, and relations $\overline{R}_i$ on $\overline{V}$ containing information about the original R_i. Such $(\overline{V}, \overline{R}_i)_{1 \leqslant i \leqslant k}$ is called a **blockmodel** for $(V, R_i)_{1 \leqslant i \leqslant k}$.

Basics of blockmodel analysis

Definition(ish) The aim of **blockmodel analysis** is to identify a partition $\overline{V}$ of V whose elements are **positions**, and relations $\overline{R}_i$ on $\overline{V}$ containing information about the original R_i. Such $(\overline{V}, \overline{R}_i)_{1 \leqslant i \leqslant k}$ is called a **blockmodel** for $(V, R_i)_{1 \leqslant i \leqslant k}$.

Blockmodels are usually formed by considering one of three types of equivalence relations encoding which actors are similar: **structural** equivalence, **automorphic** equivalence, and **regular** equivalence.

Structural and automorphic equivalence

Definition Vertices $u, v \in V$ are **structurally equivalent** if for every $w \in V$ and $1 \leqslant i \leqslant k$:

$$[u\ R_i\ w \iff v\ R_i\ w] \text{ and } [w\ R_i\ u \iff v\ R_i\ u]$$

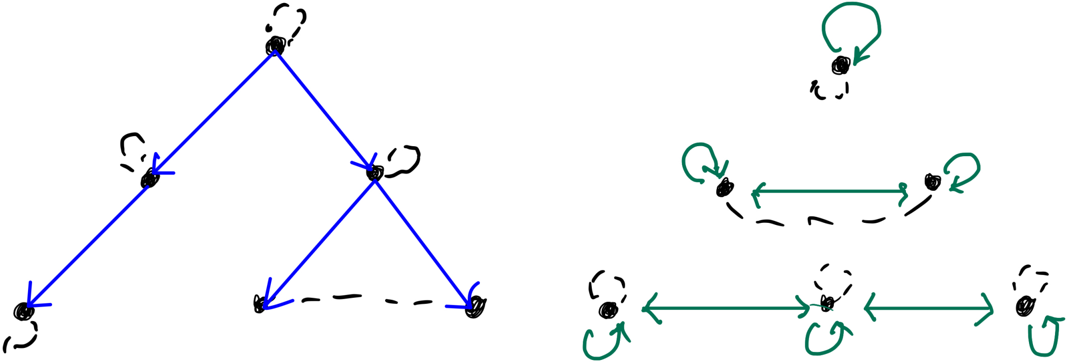
That is, if they have the same ties to all other vertices. Vertices $u, v \in V$ are **automorphically equivalent** if there is a k-graph automorphism f such that

$$f(u) = v$$

.

Structural equivalence example

Taken apart into 1-graphs:



Regular equivalence

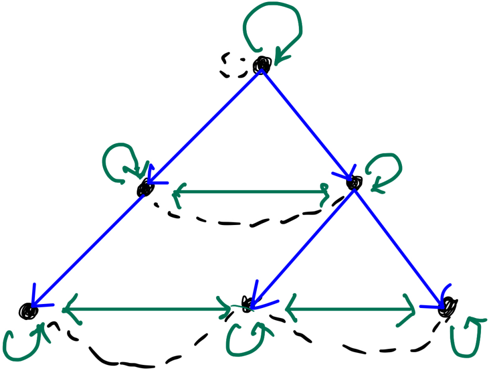
Definition An equivalence relation $B \subseteq V \times V$ is a **regular equivalence** if for every $(v, w) \in B$ and $1 \leqslant i \leqslant k$:

- If $v \; R_i \; u$, then there is x such that $w \; R_i \; x$ and $(u, x) \in B$, and
- If $u \; R_i \; v$, then there is x such that $x \; R_i \; w$ and $(u, x) \in B$.

Under a regular equivalence, *equivalent actors are equivalently related to equivalent actors*.

More general than the other two

Regular equivalence example



Bisimulations

Regular equivalences are (essentially) the same as what is called a bisimulation equivalence, well-known in modal logic and many parts of theoretical computer science

Bisimulations

Regular equivalences are (essentially) the same as what is called a bisimulation equivalence, well-known in modal logic and many parts of theoretical computer science

Definition A **bisimulation** between k-graphs (V, R_i) and (V', R_i') is a relation $B \subseteq V \times V'$ such that for all $(v, v') \in B$ and $1 \leqslant i \leqslant k$:

- If $v \; R_i \; u$, then there is $u' \in V'$ such that $v' \; R_i' \; u'$ and $(u, u') \in B$, and
- If $v' \; R_i' \; u'$, then there is $u \in V$ such that $v \; R_i \; u$ and $(u, u') \in B$

If B is an equivalence relation as well, we say it is a **bisimulation equivalence**.

Bisimulations

Regular equivalences are (essentially) the same as what is called a bisimulation equivalence, well-known in modal logic and many parts of theoretical computer science

Definition A **bisimulation** between k-graphs (V, R_i) and (V', R'_i) is a relation $B \subseteq V \times V'$ such that for all $(v, v') \in B$ and $1 \leqslant i \leqslant k$:

- If $v\ R_i\ u$, then there is $u' \in V'$ such that $v'\ R'_i\ u'$ and $(u, u') \in B$, and
- If $v'\ R'_i\ u'$, then there is $u \in V$ such that $v\ R_i\ u$ and $(u, u') \in B$

If B is an equivalence relation as well, we say it is a **bisimulation equivalence**.

A relation $B \subseteq V \times V$ is a regular equivalence iff it is a bisimulation equivalence on both (V, R_i) and (V, R_i^{op})

Positional reductions

Definition A **positional reduction** of a k-graph (V, R_i) consists of a k-graph $(\overline{V}, \overline{R_i})$ and a surjective function $\phi : V \to \overline{V}$ such that, for each i,

- If $v\ R_i\ u$, then $\phi(v)\ \overline{R}_i\ \phi(u)$ (ϕ is a k-graph homomorphism)
- If $\phi(v)\ \overline{R}_i\ u'$, then there is $u \in V$ such that $v\ R_i\ u$ and $\phi(u) = u'$ (ϕ is locally surjective)

Positional reductions are precisely the blockmodels obtained through bisimulation equivalences

Questions

This is all well and good, but how can we extend this properly to richer, possibly higher-order structures?

Part II

Blockmodel analysis through coalgebra

Universal coalgebra

Bisimulations are a central object of study in the theory of universal coalgebra

Universal coalgebra

Bisimulations are a central object of study in the theory of universal coalgebra

Definition Let T be a functor $\mathbf{C} \to \mathbf{C}$. A **$T$-coalgebra** is a pair $\mathbf{X} = (X, X \xrightarrow{\xi} TX)$ where X is an object of $\mathbf{C}$ and ξ is a morphism in $\mathbf{C}$. A **k-colored T-coalgebra** (X, ξ_i) consists of an object X of $\mathbf{C}$ with a k-indexed family of morphisms $X \xrightarrow{\xi_i} TX$.

You can think of these as generalized transition systems.

A **k-colored T-coalgebra morphism** $f : \mathbf{X} \to \mathbf{X}'$ is a morphism $f : X \to X'$ in $\mathbf{C}$ such that for all i:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle \xi_i}\downarrow & & \downarrow{\scriptstyle \xi_i'} \\ TX & \xrightarrow[Tf]{} & TX' \end{array}$$

k-graphs as coalgebras

Writing $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ for the (covariant) powerset functor:

$$\mathcal{P}(V \times V) \cong (V \to \mathcal{P}V)$$

So k-graphs (V, R_i) are k-colored $\mathcal{P}$-coalgebras

The k-colored $\mathcal{P}$-coalgebra morphisms are almost positional reductions: they are potentially non-surjective, but locally surjective k-graph homomorphisms. Surjective $\mathcal{P}$-coalgebra morphisms are precisely positional reductions.

Social systems as coalgebras

Example

Let $\mathbb{H} = (H, \oplus, \otimes, e)$ be a 'rg' / 'hemiring': $(H, \oplus, e)$ is a commutative monoid, $(H, \otimes)$ is a semigroup with absorbing element e, and $\otimes$ distributes over $\oplus$. The **$\mathbb{H}$-valuation functor**[1] $\mathbb{H}_\omega : \mathbf{Set} \to \mathbf{Set}$ is defined as

$$\mathbb{H}_\omega X = \{r : X \to H \mid r(x) \neq e \text{ for finitely many } x\}$$

$$\mathbb{H}_\omega(X \xrightarrow{f} X')(r) = x' \mapsto \bigoplus_{x \in f^{-1}(x')} r(x)$$

[1] Terminology from Bonchi et al. (2011)

Social systems as coalgebras

Example

Let $\mathbb{H} = (H, \oplus, \otimes, e)$ be a 'rg' / 'hemiring': $(H, \oplus, e)$ is a commutative monoid, $(H, \otimes)$ is a semigroup with absorbing element e, and $\otimes$ distributes over $\oplus$. The **$\mathbb{H}$-valuation functor**[1] $\mathbb{H}_\omega : \mathbf{Set} \to \mathbf{Set}$ is defined as

$$\mathbb{H}_\omega X = \{r : X \to H \mid r(x) \neq e \text{ for finitely many } x\}$$

$$\mathbb{H}_\omega(X \xrightarrow{f} X')(r) = x' \mapsto \bigoplus_{x \in f^{-1}(x')} r(x)$$

k-colored $\mathbb{H}_\omega$-coalgebras are 'k-colored graphs weighted in $\mathbb{H}$', or (essentially) square matrices valued in $\mathbb{H}^k$

[1] Terminology from Bonchi et al. (2011)

Social systems as coalgebras

For the two-element Boolean algebra 2, we get $2_\omega \cong \mathcal{P}_\omega$ with $\mathcal{P}_\omega$the finitary powerset functor, so for all practical purposes, k-colored 2_ω-coalgebras are k-graphs

Other choices of $\mathbb{H}$ give us k-graphs with richer structure, encoding e.g. the strength of ties between actors, or conditions on there being a tie between them

Social systems as coalgebras

For the two-element Boolean algebra 2, we get $2_\omega \cong \mathcal{P}_\omega$ with $\mathcal{P}_\omega$the finitary powerset functor, so for all practical purposes, k-colored 2_ω-coalgebras are k-graphs

Other choices of $\mathbb{H}$ give us k-graphs with richer structure, encoding e.g. the strength of ties between actors, or conditions on there being a tie between them

Coalgebras for the composition $\mathcal{P}\mathcal{P}$ are directed hypergraphs: a set V together with a set $E \subseteq V \times \mathcal{P}V$ of hyperedges with a single 'head'

Bisimulations on coalgebras

Many notions of bisimulations in universal coalgebra, but the most primitive (called Aczel-Mendler bisimulation) considers bisimulations to be '**the**' relations of coalgebras.

Bisimulations on coalgebras

Many notions of bisimulations in universal coalgebra, but the most primitive (called Aczel-Mendler bisimulation) considers bisimulations to be '**the**' relations of coalgebras. In the case where $\mathbf{C} = \mathbf{Set}$:

Definition For a k-colored T-coalgebra $\mathbf{X}$, an equivalence relation $B \subseteq X \times X$ is a **T-bisimulation equivalence** on $\mathbf{X}$ if there exists k-colored T-coalgebra structure $\beta_i : B \to TB$ making B into a span on $\mathbf{X}$, i.e. such that for all i:

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & B & \xrightarrow{\pi'} & X \\ \Big\downarrow{\scriptstyle \xi_i} & & \Big\downarrow{\scriptstyle \beta_i} & & \Big\downarrow{\scriptstyle \xi_i} \\ TX & \xleftarrow[T\pi]{} & TB & \xrightarrow[T\pi']{} & TX \end{array}$$

Bisimulations on coalgebras

Having bisimulations being spans of coalgebras allows us to nicely do blockmodel analysis: the blockmodel obtained from a T-bisimulation equivalence $B \subseteq X \times X$ can be computed as the coequalizer of B as a span in the category of k-colored T-coalgebras. (And colimits of coalgebras can be very concretely computed from colimits in the underlying category!)

Plus, (analogous to the first isomorphism theorem of algebra), we have that surjective T-coalgebra morphisms (generalizing positional reductions) out of a T-coalgebra $\mathbf{X}$ are precisely the results of taking coequalizers of bisimulation equivalences on $\mathbf{X}$.

Bisimulations on coalgebras

We can concretely characterize $\mathbb{H}_\omega$-bisimulation equivalences. They are equivalence relations $B \subseteq X \times X$ such that for all $(x, x') \in B$, $1 \leqslant i \leqslant k$, and B-equivalence classes $U \subseteq X$:

$$\bigoplus_{u \in U} \xi_i(x)(u) = \bigoplus_{u \in U} \xi_i(x')(u)$$

Bisimulations on coalgebras

We can concretely characterize $\mathbb{H}_\omega$-bisimulation equivalences. They are equivalence relations $B \subseteq X \times X$ such that for all $(x, x') \in B$, $1 \leqslant i \leqslant k$, and B-equivalence classes $U \subseteq X$:

$$\bigoplus_{u \in U} \xi_i(x)(u) = \bigoplus_{u \in U} \xi_i(x')(u)$$

A bisimulation equivalence on a $\mathcal{P}\mathcal{P}$-coalgebra / directed hypergraph $\mathbf{X}$ is an equivalence relation B such that for all $(x, x') \in B$:

- If (x, U) (with $U \subseteq X$) is a hyperedge starting from x, then there exists a hyperedge (x', U') such that
  - for all $u \in U$ there is $u' \in U'$ with $(u, u') \in B$,
  - for all $u' \in U'$ there is $u \in U$ with $(u, u') \in B$,
- If (x', U') (with $U' \subseteq X$) is a hyperedge starting from x', then there exists a hyperedge (x, U) such that
  - for all $u \in U$ there is $u' \in U'$ with $(u, u') \in B$,
  - for all $u' \in U'$ there is $u \in U$ with $(u, u') \in B$,

Part III

Relating blockmodel and role analysis through coalgebra

Role analysis on k-graphs

In role analysis, we study **social roles**: patterns of ties, or compound ties, between actors or positions.

Role analysis on k-graphs

In role analysis, we study **social roles**: patterns of ties, or compound ties, between actors or positions.

Definition The **semigroup of roles** in a k-graph (V, R_i) is the semigroup generated by the set $\{R_1, \dots, R_k\}$ under composition of relations. Denote it $\mathrm{Role}(V)$. The elements of $\mathrm{Role}(V)$ are called **roles** or **compound ties**.

A **role reduction** for (V, R_i) consists of a semigroup S and a surjective semigroup homomorphism $\mathrm{Role}(V) \to S$.

Functoriality of role analysis on k-graphs

Let $\mathbf{Graph}^k_{\mathbf{Surj}}$ denote the category of k-graphs and positional reductions (i.e. blockmodels obtained from bisimulation equivalences), and let $\mathbf{SemiGroup}_{\mathbf{Surj}}$ denote the category of semigroups and surjective homomorphisms.

Theorem (Otter & Porter, 2020)

The assignment of the semigroup of roles induces a functor

$$\mathrm{Role} : \mathbf{Graph}^k_{\mathbf{Surj}} \to \mathbf{SemiGroup}_{\mathbf{Surj}}.$$

The theorem tells us that every positional reduction induces a canonical role reduction, and these behave well under ‘nesting’.

Composition of social systems through coalgebra

The coalgebraic framework allows us to also naturally approach the associative composition of more general social systems.

Composition of social systems through coalgebra

The coalgebraic framework allows us to also naturally approach the associative composition of more general social systems.

If the functor T has the structure of a monad, then T-coalgebra structures on an object X are endomorphisms on X in the Kleisli category of T (denoted $\mathbf{C}_T$). Taking the powerset monad, we get that binary relations on X are the same thing as elements of the endomorphism monoid $\mathbf{Set}_{\mathcal{P}}(X, X)$.

Well-known fact Composition in $\mathbf{Set}_{\mathcal{P}}$ is composition of relations.

So one way to compose social systems is by describing them as coalgebras for a monad!

Functorial role analysis for coalgebras

Definition Let $\mathbb{T}$ be a monad on $\mathbf{C}$, and $\mathbf{X}$ a k-colored T-coalgebra . The **semigroup of $\mathbb{T}$-roles** in $\mathbf{X}$ is the subsemigroup of $\mathbf{C}_{\mathbb{T}}(X, X)$ generated by $\{X \xrightarrow{\xi_i} TX \mid 1 \leqslant i \leqslant k\}$. Denote it $\mathrm{Role}_{\mathbb{T}}(\mathbf{X})$.

Functorial role analysis for coalgebras

Definition Let $\mathbb{T}$ be a monad on $\mathbf{C}$, and $\mathbf{X}$ a k-colored T-coalgebra . The **semigroup of $\mathbb{T}$-roles** in $\mathbf{X}$ is the subsemigroup of $\mathbf{C}_{\mathbb{T}}(X, X)$ generated by $\{X \xrightarrow{\xi_i} TX \mid 1 \leqslant i \leqslant k\}$. Denote it $\mathrm{Role}_{\mathbb{T}}(\mathbf{X})$.

Let $T\mathbf{Coalg}^k_{\mathbf{Surj}}$ denote the category whose objects are k-colored T-coalgebras and whose morphisms are k-colored T-coalgebra morphisms that are epis in $\mathbf{C}$ (generalizing positional reductions).

Theorem

The assignment of the semigroup of $\mathbb{T}$-roles extends to a functor

$$\mathrm{Role}_{\mathbb{T}} : \mathbb{T}\mathbf{Coalg}^k_{\mathbf{Surj}} \to \mathbf{SemiGroup}_{\mathbf{Surj}}.$$

Taking $\mathbb{T}$ to be $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ recovers Otter & Porter's theorem.

Role analysis for coalgebras

We actually don't need a monad - a semimonad suffices, i.e. an endofunctor T with an associative multiplication $\mu : TT \Rightarrow T$.

We can equip $\mathbb{H}_\omega$ with semimonad structure by putting

$$\mu_X(W \in \mathbb{H}_\omega \mathbb{H}_\omega X) = x \in X \mapsto \bigoplus_{r \in \mathbb{H}_\omega X} (W(r) \otimes r(x))$$

Considering $\mathbb{H}_\omega$-coalgebras as matrices, the composition given by this μ is essentially matrix multiplication.

In case $\mathbb{H} = 2$, this is the usual multiplication on the (finitary) powerset functor

Role analysis for coalgebras

Denoting the multiplication on $\mathcal{P}$ by ν, we have at least two multiplications on $\mathcal{PP}$:

$$\mu^1_X = \mathcal{P}\nu_X \circ \mathcal{PP}\nu_X,\ \mu^2_X = \nu_{\mathcal{P}X} \circ \nu_{\mathcal{PP}X}$$

Here it becomes important that we work with semimonads: by Klin & Salamanca (2018), $\mathcal{PP}$ does not admit any monad structure.

Wrapping up

There's still a lot to do here:

- Apply and interpret the analyses developed here on real world data
- Consider more nuanced notions of similarity between actors: maybe two actors are only similar to some extent. There is a theory of coalgebra developed over metric-enriched categories, in which we can reason about the behavioural distance of actors
- We are not only interested in directed hypergraphs, but also e.g. simplicial complexes. Using locally surjective homomorphisms, simplicial complexes form a full subcategory of $\mathcal{P}\mathcal{P}$-coalgebras. Can we develop associative and functorial role analysis for them?
- The functoriality theorem can be stated much more generally in terms of so-called semipromonads: semimonads in the bicategory of profunctors. Can we fit other composition operations (not arising from semimonads) into this framework?