Coalgebraic Analysis of Social Systems

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Social network analysis

Social scientists often study social structure from the perspective of individual behaviour

Methods like **social network analysis** take a structural approach to social science, focusing on the connections between individuals

Multirelational graphs

The object of study: **multirelational graphs**

- Nodes are social actors – e.g. employees at a firm, family members, academics
- Edges are relationships – e.g. ‘supervising’, ‘siblinghood’, ‘coauthorship’

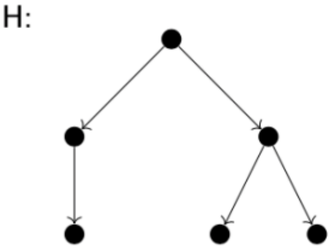
Multirelational graphs, formally

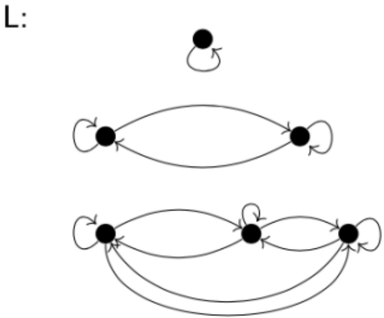
Definition

Given a set A of relation types, a **multirelational graph** consists of a set V of vertices and binary relations $R_a \subseteq V \times V$ for each type $a \in A$.

Example

Vertices are employees, H is 'supervisor of', L is 'being able to sub in for'.

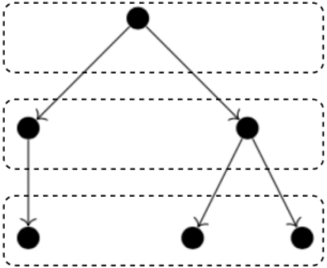


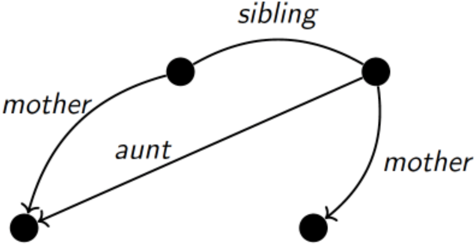


Positions and roles

Social positions: collections of actors who are similar in their relationships to others

Social roles: patterns of relationships among actors or positions





Role analysis

In **role analysis**, we study all possible compound relations (i.e. roles) in order to find or impose equalities between them.

Definition

Given a multirelational graph $G = (V, R_a)_{a \in A}$, its **semigroup of roles** is the semigroup S_G generated by $\{R_a\}_{a \in A}$ under relational composition

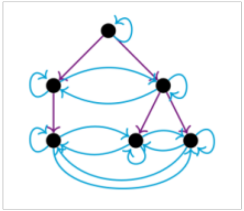
The **goal** of role analysis is then to study homomorphic reductions

$$S_G \twoheadrightarrow S$$

Role analysis, example

$\xrightarrow{H} = \longrightarrow$

$\xrightarrow{L} = \longrightarrow$



$\circ$	H	L	HL	HH
H	HH	HL	HH	0
L	HL	L	HL	HH
HL	HH	HL	HH	0
HH	0	HH	0	0

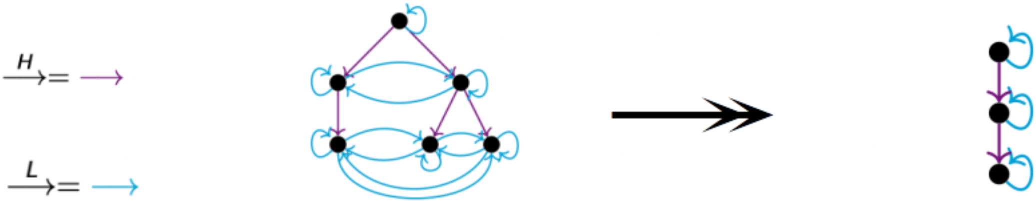
$\circ$	H	id	HH
H	HH	H	0
id	H	id	HH
HH	0	HH	0

Quotient by congruence relation obtained by setting L=id

Positional analysis

In **positional analysis**, we study multirelational graphs at different resolutions via blockmodels with positions as vertices

multirelational graph $\twoheadrightarrow$ blockmodel



Equivalence relations for positions

Structural equivalence: Two actors are equivalent if they have exactly the same neighbours for all relation types

Automorphic equivalence: Two actors are equivalent if they are identified by some automorphism of the graph

Regular equivalence:

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Structural equivalence: Two actors are equivalent if they have exactly the same neighbours for all relation types

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Regular equivalence: "Two actors are equivalent if they are equally connected to equivalent actors" Bisimulation! More on that later...

Question #1: Relating the analyses

$$\begin{tikzcd}
G = (V, R_a)_{a\in A} \arrow[r, mapsto] \arrow[d, two heads] & S_G \arrow[d, dashed, two heads, "??"] \\
G' = (V', R'_a)_{a\in A} \arrow[r, mapsto] & S_{G'}
\end{tikzcd}$$

Can we complete this diagram?

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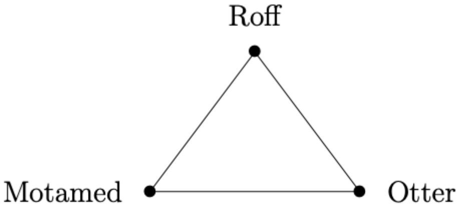
Can we complete this diagram?

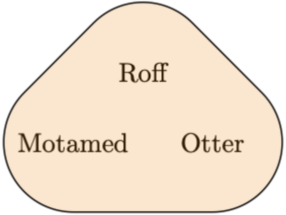
Theorem (Otter & Porter 2020)

The assignment $G \mapsto S_G$ extends to a functor from the category of graphs with surjective functional regular equivalences, to the category of semigroups with surjective homomorphisms.

Question #2: Higher-order relations

Social systems contain far more information than just binary relations





What are the right equivalence relations for positional analysis?
How does one associatively compose higher-order relations for role analysis?

AMS MRC 2022 on Applied Category Theory

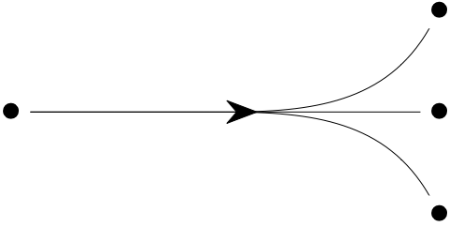
Coalgebra!

For graphs ($\mathcal{P}$-coalgebras), we do positional analysis via regular equivalences ($\mathcal{P}$-bisimulation equivalences)

Coalgebra!

For graphs ($\mathcal{P}$-coalgebras), we do positional analysis via regular equivalences ($\mathcal{P}$-bisimulation equivalences)

Directed hypergraphs (with single-vertex tails) are the $\mathcal{P}\mathcal{P}$-coalgebras $\rightsquigarrow$ we define regular equivalences as $\mathcal{P}\mathcal{P}$-bisimulation equivalences



Role analysis in coalgebra

Using the monad structure on $\mathcal{P}$, the relational composition used in the semigroup of roles is exactly composition of endomorphisms in the Kleisli category

So we just use the monad structure on $\mathcal{P}\mathcal{P}$ to do role analysis with a guaranteed associative composition . . . **all good!**

Uh-oh!

Iterated Covariant Powerset is not a Monad[1]

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But actually...

In 2018 John Baez asked on the n-Category Cafe...

Question. Does there exist an associative multiplication $m\colon P^2P^2 \Rightarrow P^2$? In other words, is there a natural transformation $m\colon P^2P^2 \Rightarrow P^2$ such that

$$P^2P^2P^2 \xRightarrow{mP^2} P^2P^2 \overset{m}{\Rightarrow} P^2$$

equals

$$P^2P^2P^2 \xRightarrow{P^2m} P^2P^2 \overset{m}{\Rightarrow} P^2.$$

Greg Egan answered: Yes! There are at least two:

$$\mu_1 : \mathcal{PPPP} \xRightarrow{\mu\mathcal{PP}} \mathcal{PPP} \xRightarrow{\mu\mathcal{P}} \mathcal{PP} \quad \text{and} \quad \mu_2 : \mathcal{PPPP} \xRightarrow{\mathcal{PP}\mu} \mathcal{PPP} \xRightarrow{\mathcal{P}\mu} \mathcal{PP}$$

where μ is the multiplication of the monad $\mathcal{P}$.

Putting it together

Given such a semimonad T and T-coalgebras $\tau = (\tau_a : V \to TV)_{a \in A}$, the **semigroup of roles** S_τ is the subsemigroup of the endomorphism semigroup of V in the Kleisli semicategory of T, generated by $(\tau_a)_{a \in A}$

Theorem

Let $T : \mathbf{C} \to \mathbf{C}$ be a semimonad. Then the assignment $\tau \mapsto S_\tau$ extends to a functor from the category of A-indexed T-coalgebras and epimorphic homomorphisms to the category of semigroups and surjective homomorphisms

A teaser!

The functoriality theorem lives at the level of (enriched) **semipromonads** (semimonads in Prof):

- A collection of objects (e.g. sets)
- Tight morphisms $X \to Y$ with identities and composition (e.g. functions)
- Loose morphisms $X \rightsquigarrow Y$ that compose with themselves and tight morphisms, but without identities (e.g. morphisms in Kleisli semicategory of $\mathcal{P}\mathcal{P}$)

Allows for functorial role & positional analysis for more general higher-order structures, and shows that role analysis for graphs is even functorial when looking at role quantales

Future work and open problems

Practically useable techniques and software!

Approximate similarity for positional analysis $\rightsquigarrow$ behavioural metrics?

Analyzing the full lattice of bisimulations